

# An $H^m$ -conforming spectral element method on multi-dimensional domain and its application to transmission eigenvalues

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**Abstract :** In this paper we develop an  $H^m$ -conforming ( $m \geq 1$ ) spectral element method on multi-dimensional domain associated with the partition into multi-dimensional rectangles. We construct a set of basis functions on the interval  $[-1, 1]$  that is made up of the generalized Jacobi polynomials (GJPs) and the nodal basis functions. So the basis functions on multi-dimensional rectangles consist of the tensorial product of the basis functions on the interval  $[-1, 1]$ . Then we construct the spectral element interpolation operator and prove the associated interpolation error estimates. Finally we apply the  $H^2$ -conforming spectral element method to the Helmholtz transmission eigenvalues that is a hot topic in the field engineering and mathematics.

**Keywords :** Spectral element method, Multi-dimensional domain, Interpolation error estimates, Transmission eigenvalues

## 1 Introduction

Spectral method is an efficient method in scientific and engineering computations which can provide superior accuracy for the solution of partial differential equations in fluid dynamics [1, 2]. But spectral method lacks the domain flexibility. So the spectral element method is developed to overcome this defect. Up to now the spectral element method have attracted more and more scholars's attention. Guo and Jia [3] studied the quadrilateral spectral method and extended it to  $H^1$ -conforming spectral element method for polygons. Shen et al. [4] provided an  $H^1$ -conforming spectral element method by constructing directly the modal basis functions on the triangle while Samson et al. [5] builds a new  $H^1$ -conforming spectral element method using the basis on the triangle by

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the rectangle-triangle mapping. Yu and Guo [6] developed an  $H^2$ -conforming spectral element method with rectangular partition in two dimension.

The orthogonal Jacobi polynomials  $\widehat{J}_i^{\alpha,\beta}(\widehat{x})$  ( $\widehat{x} \in I := [-1, 1]$ ,  $\alpha, \beta > -1$ ) weighted with  $w^{\alpha,\beta}(\widehat{x}) := (1 - \widehat{x})^\alpha(1 + \widehat{x})^\beta$  are usually adopted to form the modal basis functions in spectral method and spectral element method. Shen et al. [1] extend these polynomials to the generalized case, namely the GJPs for  $\alpha, \beta \in \mathbb{R}$ . It is worth indicating an important property of GJPs that they together with their first few order derivatives vanish at the endpoints  $\pm 1$ . So Shen et al. use them as a set of basis functions in  $H_0^m(I)$  ( $m \geq 1$ ) as well as apply them to the general order PDEs. Using the GJPs one can easily construct the tensorial basis functions in  $H_0^m$  for spectral method on multi-dimensional rectangle. Canuto et al. mentioned in section 8.5 of the book [2] a type of modal boundary-adapted basis functions on  $[-1, 1]$ . They consist of the modal basis, viewed as a compact combination of Legendre polynomials, and the nodal basis (without derivatives) at  $\pm 1$  so that one can easily establish  $H^1$ -conforming spectral element approximation. Note that these modal bases are no other than the GJPs  $\widehat{J}_i^{-1,-1}(\widehat{x})$ . Using the similar way Yu and Guo [6] developed an  $H^2$ -conforming spectral element method with rectangular partition coupled with an error analysis. This motivates us to extend this situation to  $H^m$ -conforming spectral elements on multi-dimensional domain.

In this paper, we aim to develop an  $H^m$ -conforming spectral element method on multi-dimensional domain which is the same as the one in [6] for the case  $m = 2$  in two dimension. We construct a set of basis functions on the interval  $[-1, 1]$  that is made up of GJPs, which is regarded as the bubble functions, and the nodal basis functions. So the basis functions on multi-dimensional rectangular element consist of the tensorial product of the basis functions on the interval  $[-1, 1]$  by an affine mapping. Then we construct the spectral element interpolation operator and prove the associated interpolation error estimates. Finally we shall apply the  $H^2$ -conforming spectral element method presented in this paper to the Helmholtz transmission eigenvalue problem that is a quadratic eigenvalue problem arising in inverse scattering theory for an inhomogeneous medium [7, 8]. In recent years, the numerical methods of the transmission eigenvalue problem are hot topics in the field of engineering and computational mathematics (see [9, 10, 11, 12, 13, 14, 15]). Among them [10] studied the spectral methods on the rectangle. But to our knowledge the above works do not involve spectral element method with  $d$ -dimensional rectangular partition ( $d = 2, 3$ ). In this paper we adopt the  $H^2$ -conforming method builded in [15] to construct a spectral element approximation for transmission eigenvalues. Our theoretical analysis and numerical results show that the  $H^2$ -conforming spectral element method can obtain the transmission eigenvalues of high accuracy numerically.

## 2 An $H^m$ -conforming spectral element method

In this section, we shall discuss an  $H^m$ -conforming spectral element method on  $d$ -dimensional domain  $D$  ( $d \geq 1$ ). We associate  $D$  with a sequence of rectangular partitions  $\{\pi_h\}_{h>0}$  into elements  $\kappa$ . First of all, we consider the construction of the bases on one-dimensional standard interval  $I = [-1, 1]$  containing nodal bases and modal bases. Before presentation, we use the notation  $P_N(K)$  to denote the polynomial space of degree  $\leq N$  in each variable on  $K$ .

One need to construct  $2m$  nodal basis functions  $\widehat{\phi}_j(\widehat{x})$  ( $j = 0, \dots, 2m - 1$ )

for the polynomial space  $P_{2m-1}(I)$  satisfying

$$\partial_{\hat{x}}^j \hat{\phi}_i(-1) = \partial_{\hat{x}}^j \hat{\phi}_{i+m}(1) = \delta_{i,j}, (i, j = 0, \dots, m-1).$$

Then we shall construct the modal bases on  $I$  which are actually the polynomial bubble functions on  $I$ . Let  $\hat{J}_j^{\alpha,\beta}(\hat{x})$  be the Jacobi polynomials which are orthogonal with respect to the weight function  $\hat{\omega}^{\alpha,\beta}(\hat{x}) = (1-\hat{x})^\alpha(1+\hat{x})^\beta$  ( $\alpha, \beta > -1$ ) on  $I$ :

$$\int_{-1}^1 \hat{J}_i^{\alpha,\beta}(\hat{x}) \hat{J}_j^{\alpha,\beta}(\hat{x}) \omega^{\alpha,\beta}(\hat{x}) d\hat{x} = \gamma_j^{\alpha,\beta} \delta_{i,j}.$$

where  $\gamma_j^{\alpha,\beta} = \frac{2^{\alpha+\beta+1} \Gamma(j+\alpha+1) \Gamma(j+\beta+1)}{(2j+\alpha+\beta+1) j! \Gamma(j+\alpha+\beta+1)}$ . The GJPs are defined by

$$\hat{J}_j^{\alpha,\beta}(\hat{x}) = \begin{cases} (1-\hat{x})^{-\alpha}(1+\hat{x})^{-\beta} \hat{J}_{j-j_0}^{-\alpha,-\beta}(\hat{x}), & \alpha, \beta \leq -1, \\ (1-\hat{x})^{-\alpha} \hat{J}_{j-j_0}^{-\alpha,\beta}(\hat{x}), & \alpha \leq -1, \beta > -1, \\ (1+\hat{x})^{-\beta} \hat{J}_{j-j_0}^{\alpha,-\beta}(\hat{x}), & \alpha > -1, \beta \leq -1, \end{cases}$$

where  $j \geq j_0$  with  $j_0 = -(\alpha+\beta)$ ,  $-\alpha$ ,  $-\beta$  for the above three cases, respectively.

Here we fix  $\alpha = \beta = -m$  then the GJPs  $\{\hat{J}_j^{-m,-m}(\hat{x})\}_{j \geq 2m}$  satisfy

$$\int_{-1}^1 \hat{J}_i^{-m,-m}(\hat{x}) \hat{J}_j^{-m,-m}(\hat{x}) \omega^{-m,-m}(\hat{x}) d\hat{x} = \gamma_{j-2m}^{m,m} \delta_{i,j}.$$

An attractive property of GJPs is that

$$\partial_{\hat{x}}^j \hat{J}_i^{-m,-m}(\pm 1) = 0, \quad j = 0, 1, \dots, m-1, \quad i \geq 2m.$$

In addition, GJPs can be represented as a compact combination of Legendre polynomials (see Lemma 6.1 in [1] and Remark 1) which is convenient for computations. So we adopt them to set the bubble functions on  $I$

$$\hat{\phi}_j(\hat{x}) = \hat{J}_j^{-m,-m}(\hat{x}), \quad j = 2m, 2m+1, \dots, N.$$

It is known that  $\{\hat{\phi}_j\}_{j=2m}^N$  is a set of basis functions of  $P_N^0(I) \subset H_0^m(I)$  (see [1]).

Hence  $\{\hat{\phi}_j\}_{j=0}^N$  constitutes a set of basis functions of  $P_N(I)$ . Next we consider the case of an arbitrary interval  $[a, b]$ . We define

$$\phi_j(x) = \left(\frac{b-a}{2}\right)^j \hat{\phi}_j(\hat{x}) \text{ and } \phi_{j+m}(x) = \left(\frac{b-a}{2}\right)^j \hat{\phi}_{j+m}(\hat{x}) \text{ for } 0 \leq j \leq m-1$$

$$\phi_j(x) = \hat{\phi}_j(\hat{x}) \text{ for } 2m \leq j \leq N$$

in terms with the linear transformation

$$x = \frac{b-a}{2} \hat{x} + \frac{b+a}{2}$$

then  $\{\phi_j\}_{j=0}^N$  constitutes a set of basis functions of  $P_N([a, b])$ .

Now we consider the bases for the arbitrary element  $\kappa := [a_1, b_1] \times \dots \times [a_d, b_d] \subset R^d$ . A natural choice of the bases on  $\kappa$  is the tensor product of one-dimensional basis functions. We define the linear transformation:  $x_i = \frac{b_i-a_i}{2} \hat{x}_i +$

$\frac{b_i+a_i}{2}$  ( $i = 1, \dots, d$ ). Based on the previous discussion for one dimension, one can use  $\{\prod_{i=1}^d \phi_{j_i}(x_i)\}_{j_1, \dots, j_d=0}^N \subset P_N(\kappa)$  as a set of basis functions for the element  $\kappa$ . For reading conveniently, we classify these basis functions on  $\kappa$  as follows:

Nodal basis functions:  $\prod_{i=1}^d \phi_{j_i}(x_i), j_1, \dots, j_d = 0, \dots, 2m-1$ .

$q$ -face basis functions ( $1 \leq q \leq d-1$ ): For any rearranged sequence  $\{i_l\}_{l=1}^d$  of  $\{i\}_{i=1}^d$ , define  $\prod_{i=1}^d \phi_{j_i}(x_i), j_{i_1}, \dots, j_{i_{d-q}} = 0, \dots, 2m-1, j_{i_{d-q+1}}, \dots, j_{i_d} \geq 2m$ .

Element bubble basis functions:  $\prod_{i=1}^d \phi_{j_i}(x_i), j_1, \dots, j_d \geq 2m$ .

One can easily verify the  $H^m$ -conformity for basis functions between the adjacent element  $\kappa_1$  and  $\kappa_2$ . We consider only the case that  $\kappa_1$  and  $\kappa_2$  share the common  $(d-1)$ -face  $\partial\kappa_1 \cap \partial\kappa_2 = a \times [a_2, b_2] \times \dots \times [a_d, b_d]$ .

For  $s = 0, \dots, m-1$ , we have

$$\begin{aligned}\partial_{x_1}^s \phi_{j_1}|_{\kappa_1}(a) &= \partial_{x_1}^s \phi_{j_1}|_{\kappa_2}(a), \quad j_1 = 0, \dots, 2m-1, \\ \partial_{x_1}^s \phi_{j_1}|_{\kappa_1}(a) &= \partial_{x_1}^s \phi_{j_1}|_{\kappa_2}(a) = 0, \quad j_1 \geq 2m.\end{aligned}$$

For  $s = 0, \dots, m-1$  and  $i = 2, \dots, d$ , we have with  $x_i \in [a_i, b_i]$

$$\begin{aligned}\partial_{x_i}^s \phi_{j_i}|_{\kappa_1}(x_i) &= \partial_{x_i}^s \phi_{j_i}|_{\kappa_2}(x_i) = \left(\frac{2}{b_i - a_i}\right)^{s-j_i} \partial_{\hat{x}_i}^s \hat{\phi}_{j_i}(\hat{x}_i), \quad j_i = 0, \dots, m-1, \\ \partial_{x_i}^s \phi_{j_i}|_{\kappa_1}(x_i) &= \partial_{x_i}^s \phi_{j_i}|_{\kappa_2}(x_i) = \left(\frac{2}{b_i - a_i}\right)^{s-j_i+m} \partial_{\hat{x}_i}^s \hat{\phi}_{j_i}(\hat{x}_i), \quad m \leq j_i \leq 2m-1, \\ \partial_{x_i}^s \phi_{j_i}|_{\kappa_1}(x_i) &= \partial_{x_i}^s \phi_{j_i}|_{\kappa_2}(x_i) = \left(\frac{2}{b_i - a_i}\right)^s \partial_{\hat{x}_i}^s \hat{J}_{j_i}^{-m, -m}(\hat{x}_i), \quad j_i \geq 2m.\end{aligned}$$

Therefore, the bases  $\prod_{i=1}^d \phi_{j_i}(x_i), (0 \leq j_1, \dots, j_d \leq N)$  together with their derivatives of order  $\leq m-1$  are equal on  $\partial\kappa_1 \cap \partial\kappa_2$ .

In what follows, we mainly introduce some interpolation operators that will be used in the argument afterwards.

Introduce the interpolation operator  $\hat{\Pi}_i^1$  ( $1 \leq i \leq d$ ):

$$(\hat{\Pi}_i^1 \hat{v})(\hat{x}_i) = \sum_{j=0}^{m-1} ((\partial_{\hat{x}_i}^j \hat{v})(-1) \hat{\phi}_j(\hat{x}_i) + (\partial_{\hat{x}_i}^j \hat{v})(1) \hat{\phi}_{j+m}(\hat{x}_i)),$$

and the orthogonal projector  $\hat{\Pi}_i^2$  from  $H_0^m(I)$  to  $P_N^0(I) = P_N(I) \cap H_0^m(I)$ :

$$\int_{-1}^1 \partial_{\hat{x}_i}^m (\hat{\Pi}_i^2 \hat{v}(\hat{x}_i) - \hat{v}(\hat{x}_i)) \partial_{\hat{x}_i}^m \hat{v}_N(\hat{x}_i) = 0, \quad \forall \hat{v}_N \in P_N^0(I).$$

Define  $(\Pi_i^1 v)(x_i) = (\hat{\Pi}_i^1 \hat{v})(\hat{x}_i)$  and  $(\Pi_i^2 v)(x_i) = (\hat{\Pi}_i^2 \hat{v})(\hat{x}_i)$  with  $v(x_i) = \hat{v}(\hat{x}_i)$ . Then it is obvious that  $\Pi_i^2$  is an orthogonal projector from  $H_0^m([a_i, b_i])$  to  $P_N^0([a_i, b_i])$  and

$$(\Pi_i^1 v)(x_i) = \sum_{j=0}^{m-1} ((\partial_{x_i}^j v)(-1) \phi_j(x_i) + (\partial_{x_i}^j v)(1) \phi_{j+m}(x_i)).$$

Define one-dimensional interpolation operator  $\widehat{I}_i : C^{m-1}(I) \rightarrow P_N(I)$  and  $(\widehat{I}_i \widehat{v})(\widehat{x}_i) = \widehat{v}(\widehat{x}_i)$  ( $i = 1, \dots, d$ ) satisfying

$$(\widehat{I}_i \widehat{v})(\widehat{x}_i) = (\widehat{\Pi}_i^1 \widehat{v} + \widehat{\Pi}_i^2 \circ (\mathbf{I} - \widehat{\Pi}_i^1) \widehat{v})(\widehat{x}_i),$$

where  $\mathbf{I}$  is the identity operator.

Define the function  $v(x_i) = \widehat{v}(\widehat{x}_i)$  and  $I_i : C^{m-1}([a_i, b_i]) \rightarrow P_N([a_i, b_i])$  satisfying  $(I_i v)(x_i) = v(x_i), \forall x_i \in [a_i, b_i]$  and

$$(I_i v)(x_i) = (\Pi_i^1 v + \Pi_i^2 \circ (\mathbf{I} - \Pi_i^1) v)(x_i).$$

It is obvious that  $(I_i v)(x_i) = (\widehat{I}_i \widehat{v})(\widehat{x}_i)$ .

Let  $H^s(K)$  be a Sobolev space with norm  $\|\cdot\|_{s,K}$  for a given  $K \subseteq D$  and we shall omit the subscript  $K$  if  $K = D$ . Hereafter in this paper, we use the symbols  $x \lesssim y$  to mean  $x \leq Cy$  for a constant  $C$  that is independent of the mesh size and the degree of piecewise polynomial space and may be different at different occurrences. Now we start with the one-dimensional interpolation error estimates.

**Lemma 2.1.** *Assume  $\widehat{v} \in H^t(I)$  ( $m \leq t \leq N+1$ ) then there holds for  $0 \leq s \leq m$*

$$\|\widehat{I}_i \widehat{v} - \widehat{v}\|_{s,I} \lesssim (1/N)^{t-s} \|\widehat{v}\|_{t,I}.$$

*Proof.* We consider only the case of the integers  $s$  and  $t$  since the left case can be derived by the operator interpolation theory. From Theorem 6.1 in [1] we know if  $\widehat{v} \in H_0^m(I) \cap H^t(I)$  ( $t \geq m$ ) then there holds for  $0 \leq s \leq m$

$$\|\widehat{\Pi}_i^2 \widehat{v} - \widehat{v}\|_{s,I} \lesssim (1/N)^{t-s} \|\widehat{v}\|_{t,I}.$$

Note that  $\widehat{v} - \widehat{\Pi}_i^1 \widehat{v} \in H_0^m(I)$  and  $\widehat{\Pi}_i^1(\widehat{v} - \widehat{\Pi}_i^1 \widehat{v}) = 0$ . Then

$$\begin{aligned} \|\widehat{I}_i \widehat{v} - \widehat{v}\|_{s,I} &= \|\widehat{I}_i(\widehat{v} - \widehat{\Pi}_i^1 \widehat{v}) - (\widehat{v} - \widehat{\Pi}_i^1 \widehat{v})\|_{s,I} \\ &= \|\widehat{\Pi}_i^2 \circ (\mathbf{I} - \widehat{\Pi}_i^1)(\widehat{v} - \widehat{\Pi}_i^1 \widehat{v}) - (\widehat{v} - \widehat{\Pi}_i^1 \widehat{v})\|_{s,I} \\ &\leq \|\widehat{\Pi}_i^2 \circ (\mathbf{I} - \widehat{\Pi}_i^1)(\widehat{v} - \widehat{\Pi}_i^1 \widehat{v}) - (\mathbf{I} - \widehat{\Pi}_i^1)(\widehat{v} - \widehat{\Pi}_i^1 \widehat{v})\|_{s,I} \\ &\lesssim (1/N)^{t-s} \|(\mathbf{I} - \widehat{\Pi}_i^1)(\widehat{v} - \widehat{\Pi}_i^1 \widehat{v})\|_{t,I} \\ &\lesssim (1/N)^{t-s} \|\widehat{v}\|_{t,I}. \end{aligned}$$

This concludes the proof.  $\square$

Define the  $d$ -dimensional interpolation operator  $\widehat{\mathbf{I}}_N : C^{m-1}(I^d) \rightarrow P_N(I^d)$  as

$$\widehat{\mathbf{I}}_N = \widehat{I}_1 \circ \dots \circ \widehat{I}_d.$$

One can easily verify that  $(\widehat{\mathbf{I}}_N \widehat{v})(\widehat{\mathbf{x}}) = \widehat{v}(\widehat{\mathbf{x}})$  holds for any  $\widehat{v}(\widehat{\mathbf{x}}) \in P_N(I^d)$ .

**Lemma 2.2.** *For any  $\widehat{v} \in H^t(I^d)$  with  $md \leq t \leq N+1$*

$$\|\widehat{\mathbf{I}}_N \widehat{v} - \widehat{v}\|_{s,I^d} \lesssim (1/N)^{t-s} \|\widehat{v}\|_{t,I^d}, \quad 0 \leq s \leq m.$$

*Proof.* Similar to Lemma 2.1 we only consider the case when both  $s$  and  $t$  are integers. Let  $d$  nonnegative integers  $\alpha_1, \dots, \alpha_d$  satisfy  $\sum_{i=1}^d \alpha_i = s$ . We obtain from Lemma 2.1 that

$$\|\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} (\widehat{\mathbf{I}}_N \widehat{v} - \widehat{v})\|_{0,I^d} \lesssim (1/N)^{t-s} \|\widehat{v}\|_{t,I^d},$$

and

$$\begin{aligned} & \|\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_d}^{\alpha_d} (\widehat{I}_1 - \mathbf{I})(\widehat{I}_2 \circ \dots \circ \widehat{I}_d \widehat{v} - \widehat{v})\|_{s, I^d} \\ & \lesssim (1/N)^{m-\alpha_1} \|\partial_{x_1}^m \partial_{x_2}^{\alpha_2} \dots \partial_{x_d}^{\alpha_d} (\widehat{I}_2 \circ \dots \circ \widehat{I}_d \widehat{v} - \widehat{v})\|_{0, I^d}, \end{aligned}$$

where  $\mathbf{I}$  is the identity operator. Hence by the triangular inequality

$$\begin{aligned} & \|\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} (\widehat{\mathbf{I}}_N \widehat{v} - \widehat{v})\|_{0, I^d} \\ & \lesssim \|\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} (\widehat{I}_1 \widehat{v} - \widehat{v})\|_{0, I^d} + \|\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} (\widehat{I}_2 \circ \dots \circ \widehat{I}_d \widehat{v} - \widehat{v})\|_{0, I^d} \\ & \quad + \|\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} (\widehat{I}_1 - \mathbf{I})(\widehat{I}_2 \circ \dots \circ \widehat{I}_d \widehat{v} - \widehat{v})\|_{0, I^d} \\ & \lesssim (1/N)^{t-s} \|\widehat{v}\|_{t, I^d} + \|\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} (\widehat{I}_2 \circ \dots \circ \widehat{I}_d \widehat{v} - \widehat{v})\|_{0, I^d} \\ & \quad + (1/N)^{m-\alpha_1} \|\partial_{x_1}^m \partial_{x_2}^{\alpha_2} \dots \partial_{x_d}^{\alpha_d} (\widehat{I}_2 \circ \dots \circ \widehat{I}_d \widehat{v} - \widehat{v})\|_{0, I^d} \\ & \lesssim (1/N)^{t-s} \|\widehat{v}\|_{t, I^d} + \|\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} (\widehat{I}_3 \circ \dots \circ \widehat{I}_d \widehat{v} - \widehat{v})\|_{0, I^d} \\ & \quad + (1/N)^{m-\alpha_2} \|\partial_{x_1}^{\alpha_1} \partial_{x_2}^m \dots \partial_{x_d}^{\alpha_d} (\widehat{I}_3 \circ \dots \circ \widehat{I}_d \widehat{v} - \widehat{v})\|_{0, I^d} \\ & \quad + (1/N)^{m-\alpha_1} \|\partial_{x_1}^m \partial_{x_2}^{\alpha_2} \dots \partial_{x_d}^{\alpha_d} (\widehat{I}_3 \circ \dots \circ \widehat{I}_d \widehat{v} - \widehat{v})\|_{0, I^d} \\ & \quad + (1/N)^{(m-\alpha_1)+(m-\alpha_2)} \|\partial_{x_1}^m \partial_{x_2}^m \dots \partial_{x_d}^{\alpha_d} (\widehat{I}_3 \circ \dots \circ \widehat{I}_d \widehat{v} - \widehat{v})\|_{0, I^d}. \end{aligned}$$

Repeating the above argument we get

$$\begin{aligned} & \|\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} (\widehat{\mathbf{I}}_N \widehat{v} - \widehat{v})\|_{0, I^d} \lesssim (1/N)^{t-s} \|\widehat{v}\|_{t, I^d} + \|\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} (\widehat{I}_d \widehat{v} - \widehat{v})\|_{0, I^d} \\ & \quad + \sum_{k=1}^{d-1} \sum_{i_1 < \dots < i_k \leq d-1} \left(\frac{1}{N}\right)^{(m-\alpha_{i_1})+\dots+(m-\alpha_{i_k})} \|\partial_{x_{i_1}}^m \dots \partial_{x_{i_k}}^m \partial_{x_{i_{k+1}}}^{\alpha_{i_{k+1}}} \dots \partial_{x_{i_{d-1}}}^{\alpha_{i_{d-1}}} \partial_{x_d}^{\alpha_d} \\ & \quad (\widehat{I}_d \widehat{v} - \widehat{v})\|_{0, I^d} \\ & \lesssim (1/N)^{t-s} \|\widehat{v}\|_{t, I^d}. \end{aligned}$$

This ends this proof.  $\square$

Likewise  $\mathbf{I}_N^\kappa : C^1(\kappa) \rightarrow P_N(\kappa)$  is defined as

$$\mathbf{I}_N^\kappa = I_1 \circ \dots \circ I_d.$$

In the end, we introduce the spectral element space

$$\begin{aligned} S^{N,h} &= \{v : v|_\kappa \in P_N(\kappa), \forall \kappa \in \pi_h \text{ and } \partial_{x_i}^s v \ (0 \leq i \leq d, 0 \leq s \leq m-1) \\ & \text{are continuous across } \partial\kappa_1 \cap \partial\kappa_2 \text{ for } \kappa_1, \kappa_2 \in \pi_h \text{ and } \partial\kappa_1 \cap \partial\kappa_2 \neq \emptyset\}. \end{aligned}$$

We define the spectral element interpolation operator  $\mathbf{I}_{N,h} : C^1(D) \rightarrow S^{N,h}$  as  $\mathbf{I}_{N,h}|_\kappa = \mathbf{I}_N^\kappa$  for any  $\kappa \in \pi_h$ . One can easily verify that  $(\mathbf{I}_{N,h}v)(\mathbf{x}) = v(\mathbf{x})$  holds for any  $v(\mathbf{x}) \in P_N(\kappa)$  and  $(\mathbf{I}_{N,h}v)(\mathbf{x}) = (\widehat{\mathbf{I}}_N \widehat{v})(\widehat{\mathbf{x}})$  for any  $\mathbf{x} \in \kappa$ .

Using the scaling argument, we can easily derive the interpolation error estimate on the element  $\kappa$  and the entire domain  $D$ .

**Lemma 2.3.** *For any  $v \in H^t(\kappa)$  with  $md \leq t \leq N+1$*

$$\|\mathbf{I}_{N,h}v - v\|_{s,\kappa} \lesssim (h/N)^{t-s} \|v\|_{t,\kappa}, \quad 0 \leq s \leq m.$$

**Theorem 2.4.** *For any  $v \in H^t(D)$  with  $md \leq t \leq N+1$*

$$\|\mathbf{I}_{N,h}v - v\|_{s,D} \lesssim (h/N)^{t-s} \|v\|_{t,D}, \quad 0 \leq s \leq m.$$

**Remark 1.** We consider the special case  $m = 2$ . The nodal basis functions with respect to the endpoint -1 are respectively:

$$\hat{\phi}_0(\hat{x}) = \frac{(\hat{x} - 1)^2(\hat{x} + 2)}{4}, \quad \hat{\phi}_1(\hat{x}) = \frac{(\hat{x} - 1)^2(\hat{x} + 1)}{4}.$$

Meanwhile, the nodal basis functions with respect to the endpoint 1 are respectively:

$$\hat{\phi}_2(\hat{x}) = -\frac{(\hat{x} + 1)^2(\hat{x} - 2)}{4}, \quad \hat{\phi}_3(\hat{x}) = \frac{(\hat{x} + 1)^2(\hat{x} - 1)}{4}.$$

One can easily verify that  $\hat{\phi}_0(-1) = 1$ ,  $\hat{\phi}_1'(-1) = 1$ ,  $\hat{\phi}_2(1) = 1$  and  $\hat{\phi}_3'(1) = 1$ .

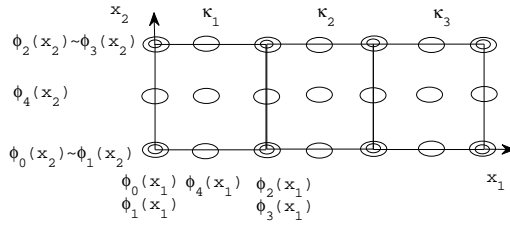
Legendre polynomials and Chebyshev polynomials are two most popular Jacobi polynomials. Now we adopt Legendre polynomials  $\{\hat{L}_j\}_{j=0}^N$  or Chebyshev polynomials  $\{\hat{T}_j\}_{j=0}^N$  to give the bubble basis functions on  $I$ . One may set

$$\hat{\phi}_j(\hat{x}) = (2j - 1)\hat{L}_{j-4}(\hat{x}) - 2(2j - 3)\hat{L}_{j-2}(\hat{x}) + (2j - 5)\hat{L}_j(\hat{x}), \quad j = 4, 5, \dots, N,$$

since  $\hat{\phi}_j(\hat{x}) = \frac{(2j-1)(2j-3)(2j-5)}{4(j-2)(j-3)}\hat{J}_j^{-2,-2}(\hat{x})$ ; another choice is

$$\hat{\phi}_j(\hat{x}) = (j - 1)\hat{T}_{j-4}(\hat{x}) - 2(j - 2)\hat{T}_{j-2}(\hat{x}) + (j - 3)\hat{T}_j(\hat{x}), \quad j = 4, 5, \dots, N.$$

**Remark 2.** One may set different polynomial degrees for each element. Figure 1 shows three element  $\kappa_1 - \kappa_3$  and the tensorial basis functions of  $P_4(\kappa_1)$  on  $\kappa_1$ . If one wants to decrease the polynomial degrees to 3 on  $\kappa_1$  and  $\kappa_2$ , one should delete the basis functions  $\phi_4(x_1)\phi_4(x_2)$  on both  $\kappa_1$  and  $\kappa_2$ ,  $\phi_2(x_1)\phi_4(x_2)$  and  $\phi_3(x_1)\phi_4(x_2)$  on the common edge of  $\kappa_1$  and  $\kappa_2$ .



**Figure 1.** The elements  $\kappa_1 - \kappa_3$  and the basis functions on  $\kappa_1$ .

**Remark 3.** The  $H^m$ -conforming spectral elements can deal with the problem with mixed boundary condition on multi-dimensional domain due to adopting the nodal basis functions at the endpoint  $\pm 1$ . Though we restrict our attention to the spectral method on rectangular domain, it can be extended to spectral method on non-rectangular domains like the way as in [3]. Likewise the mesh adopted by the spectral element method can be improved for approximating general domains better.

### 3 $H^2$ -conforming spectral elements for transmission eigenvalues

In this section, we aim to apply the  $H^2$ -conforming spectral elements in the last section to the transmission eigenvalue problem.

Consider the Helmholtz transmission eigenvalue problem: Find  $k \in \mathcal{C}$ ,  $\omega, \sigma \in L^2(D)$ ,  $\omega - \sigma \in H^2(D)$  such that

$$\Delta\omega + k^2 n(x)\omega = 0, \quad \text{in } D, \quad (3.1)$$

$$\Delta\sigma + k^2 \sigma = 0, \quad \text{in } D, \quad (3.2)$$

$$\omega - \sigma = 0, \quad \text{on } \partial D, \quad (3.3)$$

$$\frac{\partial\omega}{\partial\nu} - \frac{\partial\sigma}{\partial\nu} = 0, \quad \text{on } \partial D, \quad (3.4)$$

where  $D \subset \mathbb{R}^d$  ( $d = 2, 3$ ) is an open bounded simply connected inhomogeneous medium,  $\nu$  is the unit outward normal to  $\partial D$ .

The eigenvalue problem (3.1)-(3.4) can be stated as the classical weak formulation below (see, e.g., [16, 7, 9]): Find  $k^2 \in \mathcal{C}$ ,  $k^2 \neq 0$ , nontrivial  $u \in H_0^2(D)$  such that

$$\left(\frac{1}{n(x)-1}(\Delta u + k^2 u), \Delta v + \overline{k}^2 n(x)v\right)_0 = 0, \quad \forall v \in H_0^2(D), \quad (3.5)$$

where  $(\cdot, \cdot)_0$  is the inner product of  $L^2(D)$ . As usual, we define  $\lambda = k^2$  as the transmission eigenvalue in this paper. We suppose that the index of refraction  $n \in L^\infty(D)$  is a real valued function such that  $n - 1$  is strictly positive (strictly negative) almost everywhere in  $D$ .

Define Hilbert space  $\mathbf{H} = H_0^2(D) \times L^2(D)$  and define  $\mathbf{H}^s(K) = H^s(K) \times H^{s-2}(K)$  with norm  $\|(u, w)\|_{s,K} = \|u\|_{s,K} + \|w\|_{s-2,K}$  for a given  $K \subseteq D$ . We write  $\mathbf{H}^1 := \mathbf{H}^1(D)$  for simplicity.

Although the problem (3.5) is a quadratic eigenvalue problem, it can be linearized by introducing some variables. Using the linearized way in [15], we introduce  $w = \lambda u$ , then the problem (3.5) is equivalent to the following linear weak formulation: Find  $(\lambda, u, w) \in \mathcal{C} \times H_0^2(D) \times L^2(D)$  such that

$$\begin{aligned} \left(\frac{1}{n-1}\Delta u, \Delta v\right)_0 &= \lambda \left(\nabla\left(\frac{1}{n-1}u\right), \nabla v\right)_0 \\ &\quad + \lambda \left(\nabla u, \nabla\left(\frac{n}{n-1}v\right)\right)_0 - \lambda \left(\frac{n}{n-1}w, v\right)_0, \quad \forall v \in H_0^2(D), \end{aligned} \quad (3.6)$$

$$(w, z)_0 = \lambda(u, z)_0, \quad \forall z \in L^2(D). \quad (3.7)$$

We introduce the following sesquilinear forms

$$\begin{aligned} A((u, w), (v, z)) &= \left(\frac{1}{n-1}\Delta u, \Delta v\right)_0 + (w, z)_0, \\ B((u, w), (v, z)) &= \left(\nabla\left(\frac{1}{n-1}u\right), \nabla v\right)_0 + \left(\nabla u, \nabla\left(\frac{n}{n-1}v\right)\right)_0 - \left(\frac{n}{n-1}w, v\right)_0 + (u, z)_0, \end{aligned}$$

then (3.5) can be rewritten as the following problem: Find  $\lambda \in \mathcal{C}$ , nontrivial  $(u, w) \in \mathbf{H}$  such that

$$A((u, w), (v, z)) = \lambda B((u, w), (v, z)), \quad \forall (v, z) \in \mathbf{H}. \quad (3.8)$$



Let norm  $\|\cdot\|_A$  be induced by the inner product  $A(\cdot, \cdot)$ , then it is clear  $\|\cdot\|_A$  is equivalent to  $\|\cdot\|_{2,D}$  in  $\mathbf{H}$ .

One can easily verify that for any given  $(f, g) \in \mathbf{H}^1$ ,  $B((f, g), (v, z))$  is a continuous linear form on  $\mathbf{H}^1$ :

$$B((f, g), (v, z)) \lesssim \|(f, g)\|_{1,D} \|(v, z)\|_{1,D}, \quad \forall (v, z) \in \mathbf{H}^1. \quad (3.9)$$

Consider the dual problem of (3.8): Find  $\lambda^* \in \mathcal{C}$ , nontrivial  $(u^*, w^*) \in \mathbf{H}$  such that

$$A((v, z), (u^*, w^*)) = \overline{\lambda^*} B((v, z), (u^*, w^*)), \quad \forall (v, z) \in \mathbf{H}, \quad (3.10)$$

In order to discretize the space  $\mathbf{H}$ , we need finite element spaces to discretize  $H_0^2(D)$  and  $L^2(D)$  respectively. Since  $H_0^2(D) \subset L^2(D)$  here we can construct the spectral element space  $S_0^{N,h} := S^{N,h} \cap H_0^2(D)$  such that  $\mathbf{H}_{N,h} := S_0^{N,h} \times S_0^{N,h} \subset H_0^2(D) \times L^2(D)$ .

The conforming spectral element approximation of (3.8) is given by the following: Find  $\lambda_{N,h} \in \mathcal{C}$ , nontrivial  $(u_{N,h}, w_{N,h}) \in \mathbf{H}_{N,h}$  such that

$$A((u_{N,h}, w_{N,h}), (v, z)) = \lambda_{N,h} B((u_{N,h}, w_{N,h}), (v, z)), \quad \forall (v, z) \in \mathbf{H}_{N,h}. \quad (3.11)$$

According to Theorem 2.4, we know the following error estimates hold for spectral element space. For any  $\psi \in H_0^2(D) \cap H^{2+r}(D)$  ( $0 \leq r \leq N-1$ ) there holds

$$\inf_{v \in S^{N,h}} \|\psi - v\|_s \lesssim (h/N)^{2+r-s} \|\psi\|_{2+r}, \quad s = 0, 1, 2.$$

To give the error of eigenfunction  $(u_{N,h}, w_{N,h})$  in the norm  $\|\cdot\|_{1,D}$  we need the following regularity assumption:

**R(D).** For any  $\varrho \in H^{-1}(D)$ , there exists  $\psi \in H^{2+r_1}(D)$  satisfying

$$\begin{aligned} \Delta\left(\frac{1}{n-1}\Delta\psi\right) &= \varrho, \quad \text{in } D, \\ \psi &= \frac{\partial\psi}{\partial\nu} = 0 \quad \text{on } \partial D, \end{aligned}$$

and

$$\|\psi\|_{2+r_1} \leq C_p \|\varrho\|_{-1}, \quad (3.12)$$

where  $r_1 \in (0, 1]$ ,  $C_p$  denotes the prior constant dependent on the equation and  $D$  but independent of the right-hand side  $\varrho$  of the equation.

It is easy to know that (3.12) is valid with  $r_1 = 1$  when  $n$  and  $\partial D$  are appropriately smooth. When  $D \subset R^2$  is a convex polygon, from Theorem 2 in [17] we can get  $r_1 = 1$  if  $n \in W^{2,p}(D)$ .

In this paper, let  $\lambda$  be an eigenvalue of (3.8) with the ascent  $\alpha$ . Let  $M(\lambda)$  and  $M(\lambda_{N,h})$  be the space spanned by all generalized eigenfunctions corresponding respectively to the eigenvalues  $\lambda$  and  $\lambda_{N,h}$ . As for the dual problem (3.10), the definitions of  $M^*(\lambda^*)$  are made similarly to  $M(\lambda)$ .

In what follows, to characterize the approximation of the finite element space  $\mathbf{H}_{N,h}$  to  $M(\lambda)$  and  $M^*(\lambda^*)$ , we introduce the following quantities

$$\begin{aligned} \delta_{N,h}(\lambda) &= \sup_{\substack{(v,z) \in M(\lambda) \\ \|(v,z)\|_{2,D}=1}} \inf_{(v_{N,h}, z_{N,h}) \in \mathbf{H}_{N,h}} \|(v, z) - (v_{N,h}, z_{N,h})\|_{2,D}, \\ \delta_{N,h}^*(\lambda^*) &= \sup_{\substack{(v,z) \in M^*(\lambda^*) \\ \|(v,z)\|_{2,D}=1}} \inf_{(v_{N,h}, z_{N,h}) \in \mathbf{H}_{N,h}} \|(v, z) - (v_{N,h}, z_{N,h})\|_{2,D}. \end{aligned}$$

Note that if  $M(\lambda) \subset \mathbf{H}^t(D)$  and  $M^*(\lambda^*) \subset \mathbf{H}^t(D)$  with  $t \leq N + 1$  then  $\delta_{N,h}(\lambda) \lesssim (h/N)^{t-2}$  and  $\delta_{N,h}^*(\lambda^*) \lesssim (h/N)^{t-2}$ .

Using the spectral approximation theory [18, 19], the literature [15] established the following a priori error estimates for the finite element approximation (3.11). According to [15], the following a priori error estimates are valid for the spectral elements, as well as for the spectral method as a special case.

**Theorem 3.1.** *Suppose  $n \in W^{1,\infty}(D)$  and  $\mathbf{R}(\mathbf{D})$  is valid. Let  $\lambda_{N,h}$  be an eigenvalue of the problem (3.11) that converges to  $\lambda$ . Let  $(u_{N,h}, w_{N,h}) \in M(\lambda_{N,h})$  and  $\|(u_{N,h}, w_{N,h})\|_A = 1$ , then there exists  $(u, w) \in M(\lambda)$  such that*

$$\|(u_{N,h}, w_{N,h}) - (u, w)\|_{2,D} \lesssim \delta_{N,h}(\lambda)^{1/\alpha}, \quad (3.13)$$

$$\|(u_{N,h}, w_{N,h}) - (u, w)\|_{1,D} \lesssim (h/N)^{r_1/\alpha} \delta_{N,h}(\lambda)^{1/\alpha}, \quad (3.14)$$

$$|\lambda - \lambda_{N,h}| \lesssim (\delta_{N,h}(\lambda) \delta_{N,h}^*(\lambda^*))^{1/\alpha}. \quad (3.15)$$

## 4 Numerical Experiment

In this section, we will report some numerical experiments for solving the transmission eigenvalue problem (3.8) by the  $H^2$  conforming spectral element method (SEM) on non-rectangular domain or by the spectral method (SM) on rectangular domain. Notice that the spectral scheme in [10] is based on the iterative method in [12], which is different from the one in this paper. An obvious feature of the method in [10] is using an estimated eigenvalue to initialize the iterative procedure. We consider the case when  $D$  is the unit square or the L-shaped domain in  $d$ -dimension ( $d = 2, 3$ ) and the index of refraction  $n = 16$ ,  $f_1(x), f_2(x)$  with  $f_1(x) = 8 + x_1 - x_2$ ,  $f_2(x) = 4 + e^{x_1+x_2}$ . We use uniform rectangular refinement to obtain some partitions of  $D$ . Accordingly some numerical eigenvalues on the unit squares and the L-shaped are listed in Tables 1-5. We also depict the profiles of some eigenfunctions on the L-shaped domains with  $n = 16$  (see Figure 2).

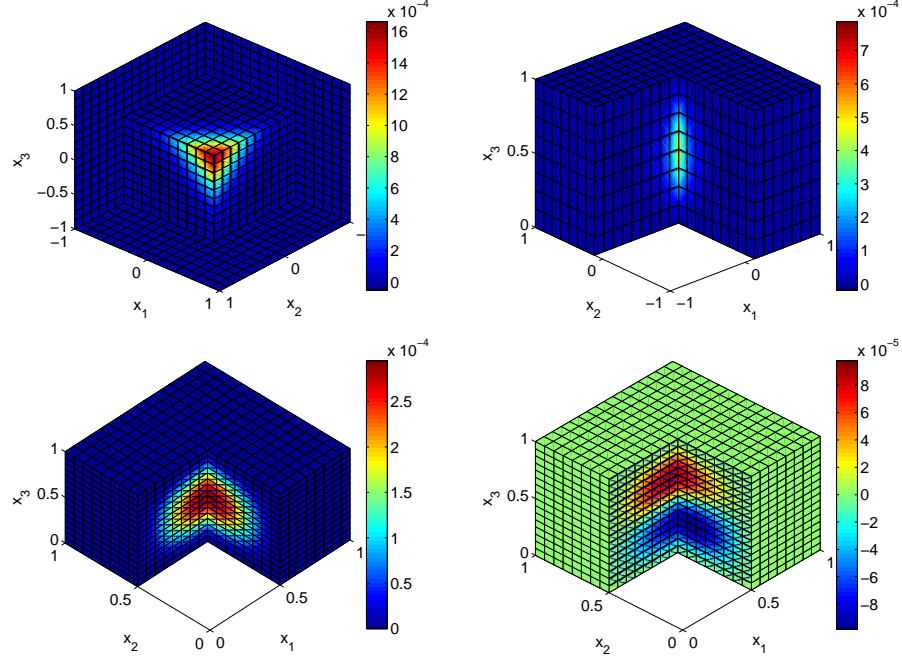
We use Matlab 2012a to solve (3.11) by the sparse solver *eigs* on a Lenovo G480 PC with 4G memory. For reading conveniently, we denote by  $k_j = \sqrt{\lambda_j}$  and  $k_{j,h} = \sqrt{\lambda_{j,h}}$  the  $j$ th eigenvalue and the  $j$ th numerical eigenvalue obtained on the space  $\mathbf{H}_h$ .

Tables 1 and 3 show that the numerical eigenvalues obtained by SM on the unit square in both two and three dimensions with different  $n$  own superior accuracy; more precisely, they achieve about eight-digit accuracy with  $N = 15$ . Whereas the numerical eigenvalues obtained by SEM on the two L-shaped domains (see Tables 2, 4-5) do not have such accuracy. This phenomenon is due to the eigenfunctions on the unit square are often smooth whereas those on the L-shaped domains have the singularities at the L-corner point (see Figure 2).

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**Figure 2.** The 1st eigenfunction for  $n = 16$  on  $(-1, 1)^3 \setminus (-1, 0)^3$  (left top), on  $((-1, 1)^2 \setminus (-1, 0]^2) \times (0, 1)$  (right top) and on  $(-1, 1)^3$  (left bottom); the 2nd eigenfunction for  $n = 16$  on  $(-1, 1)^3$  (right bottom).

Table 1: Numerical eigenvalues obtained by SM on  $(-0.5, 0.5)^2$ .

$n$	$N$	$dof$	$k_{1,h}$	$k_{2,h}, k_{3,h}$	$k_{4,h}$	$k_{5,h}$
16	15	288	1.87959117836	2.4442361007	2.86643909864	3.14011071773664
16	20	578	1.87959117345	2.4442360999	2.86643911078	3.14011071380238
16	25	968	1.87959117325	2.4442360993	2.86643910989	3.14011071380235
16	30	1458	1.87959117313	2.4442360992	2.86643910981	3.14011071380234
$n$	$N$	$dof$	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{5,h}, k_{6,h}$
$f_1$	15	288	2.8221893622	3.5386966987	3.5389915453	4.4965518722
						$\pm 0.8714816081i$
$f_1$	20	578	2.8221893415	3.5386966965	3.5389915430	4.4965519547
						$\pm 0.8714817833i$
$f_1$	25	968	2.8221893411	3.5386966953	3.5389915418	4.4965519545
						$\pm 0.8714817812i$
$f_1$	30	1458	2.8221893409	3.5386966952	3.5389915416	4.4965519545
						$\pm 0.8714817805i$
$n$	$N$	$dof$	$k_{1,h}, k_{2,h}$	$k_{3,h}$	$k_{4,h}$	$k_{5,h}$
$f_2$	15	288	4.3184549937	4.5885145655	4.6472932515	4.95760056967
			$\pm 0.6549618762i$			
$f_2$	20	578	4.3184553572	4.5885144838	4.6472932378	4.95759998805
			$\pm 0.6549618008i$			
$f_2$	25	968	4.3184553557	4.5885144805	4.6472932351	4.95759999028
			$\pm 0.6549617996i$			
$f_2$	30	1458	4.3184553554	4.5885144801	4.6472932348	4.95759999016
			$\pm 0.6549617990i$			

Table 2: Numerical eigenvalues obtained by SEM on  $(-1, 1)^2 \setminus ([0, 1] \times (-1, 0])$ .

$n$	$h$	$N$	$dof$	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{4,h}$
16	$\sqrt{2}$	15	960	1.47854	1.569782	1.705721	1.78312049
16	$\sqrt{2}$	20	1870	1.47742	1.569746	1.705408	1.78311760
16	$\sqrt{2}$	25	3080	1.47691	1.569735	1.705269	1.78311674
16	$\sqrt{2}$	30	4590	1.47665	1.569730	1.705195	1.78311641
16	$\frac{\sqrt{2}}{2}$	15	4264	1.47722	1.569741	1.705355	1.78311725
16	$\frac{\sqrt{2}}{4}$	15	17928	1.47663	1.569730	1.705189	1.78311639
16	$\frac{\sqrt{2}}{8}$	15	73480	1.47635	1.569727	1.705111	1.78311614
$n$	$h$	$N$	$dof$	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{5,h}, k_{6,h}$
$f_1$	$\sqrt{2}$	15	960	2.30499	2.395810	2.64178134	2.92613 $\pm 0.56694i$
$f_1$	$\sqrt{2}$	30	4590	2.30277	2.395702	2.64177929	2.92466 $\pm 0.56511i$
$f_1$	$\frac{\sqrt{2}}{2}$	15	4264	2.30343	2.395724	2.64177970	2.92510 $\pm 0.56566i$
$f_1$	$\frac{\sqrt{2}}{4}$	15	17928	2.30274	2.395701	2.64177927	2.92464 $\pm 0.56509i$
$f_1$	$\frac{\sqrt{2}}{8}$	15	73480	2.30241	2.395694	2.64177916	2.92442 $\pm 0.56482i$

Table 3: Numerical eigenvalues obtained by SM on  $(0, 1)^3$ .

$n$	$N$	$dof$	$k_{1,h}$	$k_{2,h}, k_{3,h}, k_{4,h}$	$k_{5,h}, k_{6,h}, k_{7,h}$	$k_{8,h}, k_{9,h}$
16	5	8	2.094055156	2.664272514	3.0661457744	3.406897998
16	10	343	2.067227464	2.584867751	2.9870636216	3.246721378
16	15	7304	2.067227678	2.584856761	2.9870431376	3.246569769
16	20	4913	2.067227671	2.584856755	2.9870431377	3.246569737
$n$	$N$	$dof$	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{4,h}$
$f_1$	5	8	3.098469114	3.865559775	3.8745648277	3.877187844
$f_1$	10	343	3.025670231	3.722083630	3.7247284048	3.724785357
$f_1$	15	7304	3.025670590	3.722061785	3.7247087240	3.724765624
$f_1$	20	4913	3.025670572	3.722061777	3.7247087161	3.724765616

Table 4: Numerical eigenvalues obtained by SEM on  $((-1, 1)^2 \setminus (-1, 0]^2) \times (0, 1)$ .

$n$	$h$	$N$	$dof$	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{4,h}$
16	$\sqrt{3}$	4	14	1.85647	1.90219	1.96071	2.07418
16	$\frac{\sqrt{3}}{2}$	4	512	1.82025	1.87691	1.93480	2.04285
16	$\frac{\sqrt{3}}{4}$	4	6800	1.80961	1.87089	1.92939	2.03857
16	$\sqrt{3}$	7	512	1.81154	1.87122	1.92966	2.03879
16	$\sqrt{3}$	8	950	1.80956	1.87078	1.92929	2.03848
16	$\sqrt{3}$	9	1584	1.80841	1.87069	1.92925	2.03845
16	$\sqrt{3}$	10	2450	1.80760	1.87064	1.92922	2.03843
$n$	$h$	$N$	$dof$	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{4,h}$
$f_1$	$\sqrt{3}$	4	14	2.69988	2.73831	2.92444	3.07209
$f_1$	$\frac{\sqrt{3}}{2}$	4	512	2.65535	2.66530	2.85574	2.98575
$f_1$	$\frac{\sqrt{3}}{4}$	4	6800	2.64344	2.64820	2.84249	2.97109
$f_1$	$\sqrt{3}$	7	512	2.64455	2.65045	2.84339	2.97245
$f_1$	$\sqrt{3}$	8	950	2.64325	2.64810	2.84230	2.97092
$f_1$	$\sqrt{3}$	9	1584	2.64269	2.64702	2.84203	2.97042
$f_1$	$\sqrt{3}$	10	2450	2.64215	2.64641	2.84186	2.97005

Table 5: Numerical eigenvalues obtained by SEM on  $(-1, 1)^3 \setminus (-1, 0)^3$ .

$n$	$h$	$N$	$dof$	$k_{1,h}$	$k_{2,h}, k_{3,h}$	$k_{4,h}, k_{5,h}$	$k_{6,h}$
16	$\sqrt{3}$	4	74	1.4993	1.5552	1.6676	1.6857
16	$\frac{\sqrt{3}}{2}$	4	1568	1.4443	1.5199	1.6443	1.6515
16	$\frac{\sqrt{3}}{4}$	4	17840	1.4277	1.5097	1.6392	1.6420
16	$\frac{\sqrt{3}}{4}$	5	45808	1.4225	1.5069	1.6388	1.6395
16	$\sqrt{3}$	6	774	1.4402	1.5161	1.6405	1.6477
16	$\sqrt{3}$	7	1568	1.4325	1.5121	1.6397	1.6442
16	$\sqrt{3}$	8	2770	1.4279	1.5097	1.6392	1.6421
16	$\sqrt{3}$	9	4464	1.4248	1.5081	1.6389	1.6406
$n$	$h$	$N$	$dof$	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{4,h}$
$f_1$	$\sqrt{3}$	4	74	2.2003	2.2307	2.2957	2.3602
$f_1$	$\frac{\sqrt{3}}{2}$	4	1568	2.1191	2.1675	2.2282	2.3227
$f_1$	$\frac{\sqrt{3}}{4}$	4	17840	2.0953	2.1523	2.2105	2.3152
$f_1$	$\frac{\sqrt{3}}{4}$	5	45808	2.0880	2.1480	2.2066	2.3143
$f_1$	$\sqrt{3}$	6	774	2.1126	2.1625	2.2197	2.3178
$f_1$	$\sqrt{3}$	7	1568	2.1020	2.1561	2.2141	2.3161
$f_1$	$\sqrt{3}$	8	2770	2.0955	2.1523	2.2105	2.3151
$f_1$	$\sqrt{3}$	9	4464	2.0913	2.1498	2.2083	2.3146

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